Tutorial Optimization

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Tutorial 17

Exercise 1

Prove, using Borel-Lebesgue definition of Compactness, that $[0, 1]$ is compact.

Exercise 2

let E be the space of real sequences (u_n) such that $| u_n | \leq 1$ for every n. For every $u \in E$, let $\|u\| = \max\{| u_n | \ n \geq 0\}$. Prove that there exists a sequence $(u(k))$ of elements of E without any convergent subsequence in $(E, \|.\|)$, i.e. $(E, \|.\|)$ is not compact.

Exercise 3 Let E be the space of real sequences (u_n). Consider the distance, on E, $d(u,v) = \sum_{n\geq 0} \frac{\min\{1,|u_n-v_n|\}}{2^n}$.

a) Prove d is well defined, and is a distance.

b) Prove that a sequence $(u(k))$ of elements of E converges to $u \in E$, where E is endowed with d, if and only if the real sequence $(u_n(k))_{k\in\mathbf{N}}$ converges to u_n for every integer n.

c) Prove that E endowed with d is compact.

Exercise 4 In the book "Introduction to dynamic optimization" (de La fuente): problem 7.17 P 88, problem 8.17 page 97, problem 8.18 page 97.

Tutorial 18

Exercise 1 Prove that in a normed space, the unit ball is compact if and only if the dimension is finite.

Exercise 2 Consider the following optimization problem, for $x_0 \geq 0$ given:

$$\max_{\forall n\geq 1, 0\leq x_t\leq f(x_{t-1})} \sum_{n\geq 1} \beta^n u(f(x_{t-1}) - x_t)$$

Here, $x_t \in \mathbf{R}$ is the capital at date t, $f(x_t) \in \mathbf{R}$ the production at date $t+1$, $f(x_{t-1})-x_t$ the consumption at date t, u a utility function, β a discount factor.

Assume f continuous and bounded from $\mathbf{R}+$ to $\mathbf{R}+$, u continuous and bounded from $\mathbf{R}$ to $\mathbf{R}_+$, $\beta \in]0,1[$.

Consider E the set of real sequences $(x_t)_{t\geq 1}$ endowed with the metric $d(x,y) = \sum_{n\geq 1} \frac{\min\{1,|x_n-y_n|\}}{2^n}$.

1) Prove that the objective function is continuous.

2) Prove that the set of constraints is closed and bounded. Prove that it is compact.

3) Prove that the previous optimization problem has at least a solution.

Tutorial 19

Exercise 1

Let $\alpha \in]0,1[$. For every subset $E \subset \mathbf{R}$ which is not bounded above, we adopt the following convention: $\sup E = +\infty$. For every mapping $f : [0,\infty[\to \mathbf{R}$, one defines the real $\|f\|$ by $\|f\| := \sup\{|e^{-2\alpha t} f(t)| : t \in [0,\infty[\}$.

Let $\mathcal{V}$ the set of continuous functions $f : [0, \infty[\to \mathbf{R}$ such htat $\|f\|$ is finite (which means $\|f\| \neq \infty$.)

1) Prove that the restriction of the function $f(t) = t$ to the set $[0, +\infty[$ is in $\mathcal{V}$.

2) Prove that $(\mathcal{V}, \|.\|)$ is a normed vector space.

3) Let(f_n) a Cauchy sequence of $(\mathcal{V}, \|.\|)$.

a) Prove that for every $t \in [0, +\infty[$, $(f_n(t))$ is a Cauchy sequence.

b) Prove that $(\mathcal{V}, \|.\|)$ is a Banach space.

Let $F : \mathbf{R} \to \mathbf{R}$ a mapping $\alpha-$Lipschitzienne, which means that for every $(t, u) \in \mathbf{R} \times \mathbf{R}$, one has $\mid F(t) - F(u) \mid \leq \alpha. \mid t - u \mid$. For every $f \in \mathcal{V}$, one defined $T(f) : [0, \infty[\to \mathbf{R}$ by

$$\forall t \in [0, \infty[: T(f)(t) = y_0 + \int_0^t F(f(s))ds.$$

5) Prove that $T(0) \in \mathcal{V}$, where 0 is the null function.

6) Prove that $\forall f, g \in \mathcal{V}, \forall t \in [0, \infty[$, one has $|T(f)(t) - T(g)(t)| \leq \frac{e^{2\alpha t}}{2}\|f - g\|$.

7) Deduce that $\forall f \in \mathcal{V}$, one has $T(f) \in \mathcal{V}$.

8) Prove that there exists a constant $\beta \in]0, 1[$ such that for every $(f, g) \in \mathcal{V} \times \mathcal{V}$, one has $\|T(f) - T(g)\| \leq \beta.\|f - g\|$.

9) Prove that there exist a unique $f \in \mathcal{V}$ such that $T(f) = f$.

Exercise 2 Let (E, d) a complete metric space and Λ a metric space. Consider a mapping $f : E \times \Lambda \to E$ such that :

- For every $x \in E$, the mapping from $\Lambda \to E$, $\lambda \mapsto f(x, \lambda)$ is continuous;

- $\exists k < 1$ such that $d(f(x, \lambda), f(y, \lambda)) \leq kd(x, y)$ for every $(\lambda, x, y) \in \Lambda \times E \times E$.

1 - Prove that for every $\lambda \in \Lambda$, there exists a unique $a_\lambda \in E$ such that $f(a_\lambda, \lambda) = a_\lambda$.

2 - Prove that the mapping $\lambda \mapsto a_\lambda$ is continuous.

Exercise 3 Let (X, d) a compact and nonempty metric space. A mapping f from X to X is if $d(f(x), f(y)) < d(x, y)$ for every $(x, y) \in X \times X$ such that $x \neq y$.

1 - Prove that a weakly contracting mapping has a unique fixed point (i.e. the equation $f(x) = x$has a unique solution).

2 - Prove that it is false if $X = \mathbf{R}$.

Tutorial 20

Exercice 1

Let X and Y two metric spaces, and $f : X \to Y$ a continuous mapping. Prove that F from X to Y defined by

$$\forall x \in X, F(x) = \{f(x)\}$$

is l.s.c. and u.s.c.

Exercice 2

Let X and Y two metric spaces, and for every $i = 1, ..., n$ let $f_i : X \to Y$ continuous mappings. Let F multivalued from X to Y defined by

$$\forall x \in X, F(x) = co\{f_i(x), i = 1, ..., n\}.$$

Prove it is l.s.c. and u.s.c.

Exercice 3

To every $p = (p_1, p_2) \in \mathbf{R}^2_+$, we associated

$$\Phi(p) = \{(x_1, x_2) \in \mathbf{R}^2, p_1.x_1 + p_2.x_2 \leq 0\}.$$

Is Φ l.s.c., u.s.c. ?

Exercice 4

Let Φ be a multivalued mapping from X to$Y$, two metric spaces. Assume Φ l.s.c.

a) Prove that $\overline{\Phi}$ is lower semi-continuous.

b) Assume Y normed space. Prove that $x \to \text{co}\Phi(x)$ is lower semi-continuous.

Exercice 5

LetΦ be a multivalued mapping from X to$Y$, two metric spaces. The space X is assumed to be compact, and Φ u.c.s., with compact values. Prove that $\Phi(X) = \cup_{x \in X} \Phi(x)$ is compact.

Tutorial 21

Exercice 1 Let X and Y two metric spaces, F a multivalued function from X to Y with compact, non empty values. Let f a continuous function from

$X \times Y$ to $\mathbf{R}$ Let $g(x) = \max_{y \in F(x)} f(x,y)$ and $M(x) = \{y \in F(x) : f(x,y) = g(x)\}$. Prove that g is continuous and M is u.s.c. with nonempty values.

Exercice 2 Prove Blackwell theorem.

Exercice 3 Let K a compact metric space. Let f a continuous function from $\mathbf{R}^d \times K$ to $\mathbf{R}$ Let $v(x) = \max_{y \in K} f(x,y)$. Assume that for every $x \in \mathbf{R}^d$, there exists a unique $y(x)$ such that $v(x) = f(x, y(x))$. Assume that for every $y \in K$, $f(.,y)$ is differentiable and $\nabla_x f$ is continuous with respect to (x,y).

Prove that v is of class C^1 and that $\nabla v(x) := \nabla_x f(x, y(x))$ for every x.

Tutorial 22

In the book "Introduction to dynamic optimization" (de La fuente): problem 8.26 P 100, proof of Theorem 1.1 p 551, problem 1.2 P 552., theorem 1.5 page 561 proof of Theorem 1.6 p 563.

Tutorial 23

In the book "Introduction to dynamic optimization" (de La fuente): problem 1.7 P 563.,problem 1.11 page 564, Theorem 1.12 page 564, problem 1.14 page 564, problem 1.17 page 564

Tutorial 24

Free tutorial for questions, explanations, ...